6.5 Positive Definite Matrices

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* This section concentrates on symmetric matrices that have **positive eigenvalues.**

* Two goals of this section:

- To find *quick tests* on a symmetric matrix that guarantee *positive eigenvalues.*
- To explain two applications of positive definiteness.

* When does $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ have $\lambda_1 > 0$ and $\lambda_2 > 0$?

6L ***The eigenvalues of A are positive if and only if $a > 0$ and $ac - b^2 > 0$.***

Ex: $A = \begin{bmatrix} 4 & 5 \\ 5 & 7 \end{bmatrix}$ has $a = 4$, $ac - b^2 = 28 - 25 = 3 > 0$

$\therefore$ A has positive eigenvalues.

6M *The eigenvalues of $A = A^T$ are positive if and only if the pivots are positive:*

$$a > 0 \quad \text{and} \quad \frac{ac - b^2}{a} > 0.$$

Ex. $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ has a negative determinant and pivot. So a **negative eigenvalue.**

Definition The matrix A is ***positive definite*** if $\mathbf{x}^{\mathrm{T}}A\mathbf{x} > 0$ for every nonzero vector:

$$\mathbf{x}^{\mathrm{T}}A\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2 > 0.$$

Example 2 This matrix A is positive definite. We test by pivots or determinants:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} \quad \text{has positive pivots and determinants (1 and 3).}$$

More directly, $\mathbf{x}^{\mathrm{T}}A\mathbf{x} = x^2 + 4xy + 7y^2$ is positive because it is a **sum of squares**:

$$\text{Rewrite} \quad x^2 + 4xy + 7y^2 \quad \text{as} \quad (x + 2y)^2 + 3y^2.$$

* First Application : Test for a Minimum

The function $f = \mathbf{x}^{\mathrm{T}}A\mathbf{x}$ *has a minimum at* $x = y = 0$ *if and only if* A *is positive definite*

Example 3 Is $f(x, y) = x^2 + 8xy + 3y^2$ everywhere positive—except at $(0, 0)$?

Solution

$$x^2 + 8xy + 3y^2 = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The matrix has $ac - b^2 = 3 - 16$. The pivots are 1 and -13.

$\therefore$ The matrix is not positive definite.

Example 4 For which numbers c is $x^2 + 8xy + cy^2$ always positive (or zero)?

Solution

$$A = \begin{bmatrix} 1 & 4 \\ 4 & c \end{bmatrix}$$

① $a = 1 > 0$

② $ac - b^2 = c - 16$

$\therefore$ For a positive definite matrix we need $c > 16$

* Second Application: The Ellipse $ax^2+2bxy+cy^2=1$

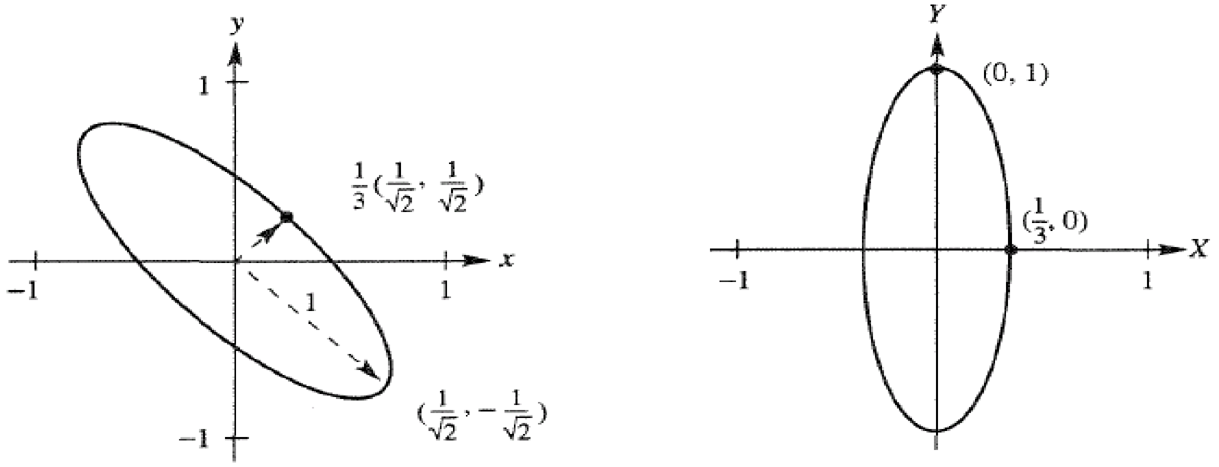

Figure 6.4 The tilted ellipse $5x^2 + 8xy + 5y^2 = 1$. Lined up it is $9X^2 + Y^2 = 1$.

These two pictures show the geometry behind the factorization $A = Q\Lambda Q^{-1} = Q\Lambda Q^{T}$:

1. The tilted ellipse is associated with A. Its equation is $\mathbf{x}^{T}A\mathbf{x} = 1$.
2. The lined-up ellipse is associated with Λ. Its equation is $\mathbf{X}^{T}\Lambda\mathbf{X} = 1$.
3. The rotation matrix from $\mathbf{x}$ to $\mathbf{X}$ that lines up the ellipse is Q.

6P Suppose $A = Q\Lambda Q^{T}$ is positive definite. The graph of $\mathbf{x}^{T}A\mathbf{x} = 1$ is an ellipse:

$$\begin{bmatrix} x & y \end{bmatrix} Q\Lambda Q^{T} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X & Y \end{bmatrix} \Lambda \begin{bmatrix} X \\ Y \end{bmatrix} = \lambda_1 X^2 + \lambda_2 Y^2 = 1.$$

The half-lengths of the axes are $1/\sqrt{\lambda_1}$ and $1/\sqrt{\lambda_2}$.

Example 7 Find the axes of this tilted ellipse $5x^2 + 8xy + 5y^2 = 1$.

Solution The function is $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1$. The matrix is $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

The eigenvalues of A are: $\lambda_1 = 9$, $\lambda_2 = 1$

The eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$A = Q \Lambda Q^T \Rightarrow$ $\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$[x \ y] \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [x \ y] \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\Rightarrow$ $5x^2 + 8xy + 5y^2 = 9\left(\frac{x+y}{\sqrt{2}}\right)^2 + 1\left(\frac{x-y}{\sqrt{2}}\right)^2$, $\frac{x+y}{\sqrt{2}} = X$ and $\frac{x-y}{\sqrt{2}} = Y$.

The ellipse becomes $9X^2 + Y^2 = 1$ The largest value of $X^2 = \frac{1}{9}$

- The point at the end of the shorter axis has $\begin{cases} X = \frac{1}{3} \\ Y = 0 \end{cases}$
- The point at the end of the major axis has $\begin{cases} X = 0 \\ Y = 1 \end{cases}$

Note: The bigger eigenvalue λ_1 gives the shorter axis of half-length $1/\sqrt{\lambda_1} = \frac{1}{3}$

The smaller eigenvalue λ_2 gives the greater length $1/\sqrt{\lambda_2} = 1$

* Summary.

6N When a 2 by 2 symmetric matrix has one of these four properties, it has them all:

1. Both of the eigenvalues are positive.
2. The 1 by 1 and 2 by 2 determinants are positive: $a > 0$ and $ac - b^2 > 0$.
3. The pivots are positive: $a > 0$ and $(ac - b^2)/a > 0$.
4. The function $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = ax^2 + 2bxy + cy^2$ is positive except at $(0, 0)$.

6O When a symmetric matrix has one of these four properties, it has them all:

1. All n ***eigenvalues*** are positive.
2. All n ***upper left determinants*** are positive.
3. All n ***pivots*** are positive.
4. $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$ is positive except at $\boldsymbol{x} = \boldsymbol{0}$. The matrix A is ***positive definite***.

■ REVIEW OF THE KEY IDEAS ■

1. Positive definite matrices have positive eigenvalues and positive pivots.
2. A quick test is given by the upper left determinants: $a > 0$ and $ac - b^2 > 0$.
3. The quadratic function $f = \boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$ then has a minimum at $\boldsymbol{x} = \boldsymbol{0}$:

$$\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = ax^2 + 2bxy + cy^2 \text{ is positive except at } (x, y) = (0, 0).$$

4. The ellipse $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 1$ has its axes along the eigenvectors of A.
5. *Coming*: $A^{\mathrm{T}}A$ is automatically positive definite if A has independent columns.